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Homotopy of Algebras*

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The purpose of this paper is to introduce a notion of homotopy of algebras. Our definition of homotopy will be reasonably simple and natural. We shall discuss only elementary properties of such homotopy, construct a homotopy category of algebras and show that there is de Rham cohomology is a functor on the homotopy category of commutative algebras. It seems that our notion of homotopy together with a related notion of paths ([*1*], [*2*]) will provide ingredients for a reasonable homotopy theory (see [*4*]) for algebras. We shall limit ourselves to associative algebras.

The basic idea behind our construction of homotopy of algebras is not new. It is a straightforward analogy of homotopy of topological spaces and can be also taken, in a sense, as some kind of deformation [*3*], [*7*]. The distinct feature of this work is the use of so called "split exact" derivations, which arises from a geometrical consideration that we are going to describe.

Two continuous maps $f_i : X' \to X$, $i = 0, 1$, are homotopic if there exist a "suitable" topological space I, two points t_0, $t_1 \in I$ and a continuous map $F : X' \times I \to X$ such that f_i is the composition $X' \to X' \times I \xrightarrow{F} X$ given by $x' \mapsto (x', t_i) \mapsto F(x', t_i)$. Customarily, I is taken to be the unit interval. There is no reason why I can not be taken to be a tree, in which there is essentially only one path between any two given points.

Let us choose a commutative ground ring K with unit element. Two morphisms of K-algebras $\phi_i : A \to A'$, $i = 0,1$, will be defined to be homotopic if there exist a "suitable" K-algebra B, two augmentations s_0 and s_1 of B and a morphism of K-algebras $\Phi : A \to A' \otimes B$ such that $(1 \otimes s_i)\,\Phi = \Phi_i$, $i = 0,1$. The algebra B should be a counterpart of a tree. We choose B to be a commutative K-algebra with unit element such that it possesses a split exact sequence of K-modules

$$0 \to K \xrightarrow{\eta} B \xrightarrow{\partial} N \to 0$$

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where $\eta = \eta_B$ is the canonical morphism, and ∂ is a derivation from B to a B-module N. Our choice of B arises from the observation that, if a tree is differentiably, say C^∞, embedded in a Euclidean space and if differentiable functions and differentials (i.e., 1-forms) on the tree are taken to be the respective restrictions of those on the Euclidean space, then every differential on the tree is an exact differential. In other words, the usual differentiation on the tree is a split exact derivation. This property characterizes a tree among graphs.

In section 1, we construct a homotopy category of algebras. In Section 2, we treat two extreme cases of algebras in relation to homotopy, namely, inmobile and contractible algebras. The triviality of the Kähler module (the A-module of universal differentials) of a commutative algebra A will be shown to be a sufficient and "nearly" necessary condition for the inmobility of A. The material in Sections 3 and 4 shows that the cohomology algebra obtained from the exterior algebra of the Kähler module of a commutative algebra A is invariant under homotopy. Such cohomology algebra is therefore trivial if A is contractible. (If K is a field of characteristic 0 and if A is graded by natural integers and is connected, then A is contractible.)

We shall use A, A', A'' to denote K-algebras; B, B_1, B_2 to denote commutative K-algebras and M, M', M'', N, N_1, N_2 to denote K-modules. We write $M \otimes N = M \otimes_K N$. All rings will possess unit element, all morphisms of rings will preserve the unit element, and all modules will be unitary.

1. Homotopy Category of Algebras

1.1. By a derivation of a commutative K-algebra A, we mean a morphism of K-modules d from A into an A-module M such that

$$d(a_1a_2) = a_2da_1 + a_1da_2\,, \forall a_1\,, a_2 \in A.$$

Definition. A derivation $\partial : B \to N$ is split exact if the sequence $0 \to K \xrightarrow{\eta} B \xrightarrow{\partial} N \to 0$ is exact and is split by an augmentation of B.

Immediate examples of split exact derivations are the usual derivations of the polynomial algebra $K[t]$ and of the formal power series algebra $K[[t]]$, in the case of K being a field of characteristic 0. In this case, even the polynomial algebra $K[x, y]$ has a split exact derivation as given in the next example.

Example. Write $B = K[x, y]$. Denote by $d : B \to \tilde{N}$ the usual derivation of B. Denote by N the quotient B-module $\tilde{N}/(ydx - xdy)$ and by $\rho : \tilde{N} \to N$ the projection. The composition $B \xrightarrow{d} \tilde{N} \xrightarrow{\rho} N$ is a derivation ∂ of B. A simple computation shows that

$$\tilde{N} = dB \oplus (ydx - xdy).$$

Thus ∂ is surjective. The exact sequence

$$0 \to K \to B \xrightarrow{\partial} N \to 0$$

is split by the augmentation s_0 of B such that $s_0 x = s_0 y = 0$.

For any given commutative groung ring K, the K-algebra $K\{t\}(K\{\{t\}\})$ of divided power polynomials (divided power series) has a split exact derivation.

Recall that the elements of $K\{t\}(K\{\{t\}\})$ are finite (infinite) formal sums

$$c_0 t^{(0)} + \cdots + c_r t^{(r)} + \cdots, \quad c_r \in K.$$

The addition and multiplication are the same as those of polynomials (power series) except that

$$t^{(r)} t^{(s)} = \binom{r+s}{r} t^{(r+s)}.$$

(See [4].)

Set $B = K\{t\}(K\{\{t\}\})$. Let s_0 be the augmentation of B such that

$$s_0(c_0 t^{(0)} + \cdots + c_r t^{(r)} + \cdots) = c_0 .$$

Let N be the free B-module generated by a single element dt. Then the derivation $\partial : B \to N$ given by

$$\partial(c_0 t^{(0)} + \cdots + c_r t^{(r)} + \cdots) = (c_1 t^{(0)} + \cdots + c_r t^{(r-1)} + \cdots)\, dt$$

is split exact.

If K is a field of characteristic $p > 0$, it can be shown that any K-algebra with a split exact derivation possesses only one augmentation. Though our notion of homotopy becomes trivial in this case, concepts of infinitesimal nature such as inmobility will remain meaningful from the point of view of split exact derivations.

Further treatment on split exact derivations (split surjective differentiations) can be found in [1] and [2].

1.2. Definition. Two morphisms of K-algebras $\phi_i : A \to A'$, $i = 0,1$, are said to be homotopic : $\phi_0 \sim \phi_1$, if there exist a commutative K-algebra B with a split exact derivation $\partial : B \to N$, a morphism of K-algebras $\Phi : A \to A' \otimes B$ and two augmentations s_0, s_1 of B such that

$$\phi_i = (1 \otimes s_i)\, \Phi, \quad i = 0,1.$$

The homotopy relation $\sim$ is reflexive and symmetrical. In order to establish its transitivity, we introduce the next definition.

DEFINITION. If $p' : A' \to K$ and $p'' : A'' \to K$ are augmentations, define

$$A' \vee A'' = A' \vee_{(p',p'')} A''$$
$$= \{(a', a'') \in A' \oplus A'' : p'a' = p''a''\}.$$

If A' and A'' are commutative and if $d' : A' \to M'$ and $d'' : A'' \to M''$ are derivations, define

$$d' \vee d'' = d' \vee_{(p',p'')} d'' : A' \vee A'' \to M' \oplus M''$$

to be the restriction of $d' \oplus d''$ on $A' \vee A''$. Then $d' \vee d''$ is a derivation. If both d' and d'' are surjective, so is $d' \vee d''$. If d' and d'' are split exact, so is $d' \vee d''$.

Define $p' \vee p''$ to be the augmentation of $A' \vee A''$ such that

$$(p' \vee p'')(a', a'') = p'a' = p''a''.$$

Observe that $A' \vee A''$ is a direct summand of $A' \oplus A''$. A direct verification yields the next assertion.

PROPOSITION. *The canonical isomorphism*

$$A \otimes (A' \oplus A'') \approx (A \otimes A') \oplus (A \otimes A'')$$

induces an isomorphism of K-algebras

$$A \otimes (A' \vee A'') \approx (A \otimes A') \vee_{(1\otimes p', 1\otimes p'')} (A \otimes A'').$$

1.3. We are going to show the transitivity of the homotopy relation.

Suppose that homotopies $\phi_0 \sim \phi_1$ and $\phi_1 \sim \phi_2$ are respectively given by $\Phi' : A \to A' \otimes B'$ and $\Phi'' : A \to A' \otimes B''$ with $\phi_0 = (1 \otimes s_0') \Phi'$,

$$\phi_1 = (1 \otimes s_1') \Phi' = (1 \otimes s_0'') \Phi'',$$

and $\phi_2 = (1 \otimes s_1'') \Phi''$.

Let $B = B' \vee B''$ and $\partial = \partial' \vee \partial''$, where $\vee = \vee_{(s_1', s_0'')}$. Let the augmentations s_0 and s_2 of B be defined respectively by the compositions

$$B \longrightarrow B' \xrightarrow{s_0'} K \qquad \text{and} \qquad B \longrightarrow B'' \xrightarrow{s_1''} K,$$

where $B \to B'$ and $B \to B''$ are projections. Define Φ to be the composition

$$A \xrightarrow{\tilde{\Phi}} (A' \otimes B') \bigvee (A' \otimes B'') \approx A' \otimes B,$$

where $\tilde{\Phi}a = (\Phi' a, \Phi'' a)$, $\forall\, a \in A$. We have $\phi_i = (1 \otimes s_i)\,\Phi$, $i = 0,2$. Hence $\phi_0 \sim \phi_2$, Q.E.D.

If $\psi' : A' \to A''$ and $\psi'' : A'' \to A'$ are morphisms of K-algebras, then $\phi_0 \sim \phi_1$ implies $\psi'\phi_0 \sim \psi'\phi_1$ and $\phi_0\psi'' \sim \phi_1\psi''$. Thus we have a homotopy category of K-algebras, whose objects are K-algebras and whose morphisms are equivalence classes of morphisms of K-algebras under the homotopy relation. Two K-algebras that are equivalent in the homotopy category are said to be of the same homotopy type.

1.4. Suspension and cofibration are two essential notions to a homotopy theory. Cofibration can be readily defined as follows: A morphism of K-algebras $f : A' \to A_1'$ is called a cofibration with respect to A if, given any commutative K-algebra B with a split exact derivation and an augmentation s_0 and given any morphisms of K-algebras $g : A \to A'$ and $\Phi_1 : A \to A_1' \otimes B$ with $fg = (1 \otimes s_0)\ \Phi_1$, there exists a morphism of K-algebras $\Phi : A \to A' \otimes B$ such that $g = (1 \otimes s_0)\,\Phi$ and $\Phi_1 = (f \otimes 1)\,\Phi$. One can even verify that such a system of cofibrations enjoys the push-out property. The notion of suspension in this frame work should be related to the notion of paths as given in [*2*] and will not be discussed here.

2. Immobility and Contractibility

2.1. Contractible K-algebras are K-algebras of the same homotopy type as K. In order to compare with immobility, we use an equivalent definition of contractibility.

Definition. A K-algebra A is contractible if there exists an augmentation p of A such that any morphism of K-algebras $\phi : A \to A'$ is homotopic to the composition $A \xrightarrow{p} K \to A'$, where $K \to A'$ is canonical.

We might define an immobile K-algebra A to be such that any given morphism of K-algebras $A \to A'$ is homotopic only to itself. However the next stronger definition seems to be more satisfactory.

Definition. An immobile K-algebra A is one which satisfies the condition that, if A' is a K-algebra, if B is a commutative K-algebra with a split exact derivation and an augmentation s_0 and if $\Phi : A \to A' \otimes B$ is a morphism of K-algebras, then Φ must be the composition

$$A = A \otimes K \xrightarrow{\varphi \otimes \eta} A' \otimes B$$

where $\varphi = (1 \otimes s_0)\, \Phi$.

Let $\varphi_i : A \to A'$, $i = 0, 1$, be homotopic. If A is immobile, then $\varphi_0 = \varphi_1$. Any two immobile algebras of the same homotopy type must be isomorphic.

2.2. THEOREM. *Let A be a commutative K-algebra. If the Kähler module $E(A) = 0$, then A is immobile.*

Proof. Let $\Phi : A \to A' \otimes B$ be such that $(1 \otimes s_0)\, \Phi = \varphi$. The composition

$$A \xrightarrow{\Phi} A' \otimes B \xrightarrow{1 \otimes \partial} A' \otimes N$$

is a derivation and can be factored through $E(A) = 0$. Consequently, $(1 \otimes \partial)\, \Phi$ is the zero morphism, i.e., $\Phi A \subset \mathrm{Ker}(1 \otimes \partial)$. On the other hand, the sequence

$$0 \longrightarrow A' \otimes K \xrightarrow{1 \otimes \eta} A' \otimes B \xrightarrow{1 \otimes \partial} A' \otimes N \longrightarrow 0$$

is split exact. We have $\Phi A \subset \mathrm{Im}(1 \otimes \eta)$, and there is a factorization of Φ :

$$A \xrightarrow{\varphi_0} A' \otimes K \xrightarrow{1 \otimes \eta} A' \otimes B.$$

Moreover $\varphi = (1 \otimes s_0)\, \Phi = (1 \otimes s_0)(1 \otimes \eta)\, \varphi_0 = \varphi_0$.
Hence $\Phi = \varphi_0 \otimes \eta = \varphi \otimes \eta$, Q.E.D.

COROLLARY. *If A is a separable algebraic extension of a field K, then the K-algebra A is immobile.*[1]

2.3. PROPOSITION. *If A is an inmobile K-algebra and if X is a derivation of A into itself, then, for any augmentation p of A and any integer $r > 0$, $pX^r : A \to K$ is always zero.*

Proof. We call for assistence from the K-algebra $K\{\{t\}\}$ of divided power series. (See 1.1.) Let $\Phi : A \to K\{\{t\}\}$ be given by

$$\Phi a = (pa)\, t^{(0)} + \cdots + (pX^r a)\, t^{(r)} + \cdots .$$

Since $X^r(aa') = \sum_{0 \leqslant i \leqslant r} \binom{r}{i}(X^{r-i}a)(X^i a')$, Φ is a morphism of K-algebras.

[1] M. E. Sweedler points out that the corollary is valid for separable K-algebras.

Let s_0 be the augmentation of $K\{\{t\}\}$ as given in 1.1. Since $(1 \otimes s_0)\, \Phi a = pa$, the immobility of A implies that Φ is the morphism

$$A = A \otimes K \xrightarrow{p \otimes \eta} K \otimes K\{\{t\}\} = K\{\{t\}\},$$

and, $\forall a \in A$, $\Phi a = pa$, which implies $pX^r a = 0$, Q.E.D.

2.4. THEOREM. *If a commutative K-algebra A' is of the same homotopy type as a commutative inmobile K-algebra A, and if $\varphi : A \to A'$ is a homotopy equivalence, then A' is contractible as an A-algebra via φ.*

Proof. Let $\psi : A' \to A$ such that $\psi\varphi \sim 1_A$ and $\varphi\psi \sim 1_{A'}$. Since $\psi\varphi = 1_A$, not only φ but also ψ can be taken as morphisms of A-algebras. We are going to show that $\varphi\psi \sim 1_{A'}$ as morphisms of A-algebras.

Let $\Phi : A' \to A' \otimes B$ be a morphism of K-algebras such that $(1 \otimes s_0)\, \Phi = 1_{A'}$ and $(1 \otimes s_1)\, \Phi = \varphi\psi$. Construct the morphism of K-algebras

$$\theta : A' \otimes B \to A' \otimes_A (A \otimes B)$$

such that $\theta(a' \otimes b) = a' \otimes_A (1 \otimes b)$. Set $\Phi' = \theta\Phi$.
Observe that the A-algebra $A \otimes B$ has a split exact derivation $1 \otimes \partial : A \otimes B \to A \otimes N$ and augmentations $s_i' = 1 \otimes s_i$, $i = 0, 1$. Observe also that

$$(1 \otimes s_i')\, \Phi' = (1 \otimes s_i')\, \theta\Phi = (1 \otimes s_i)\, \Phi.$$

It remains to show that Φ' is a morphism of A-algebras. In fact, since A is immobile, we have $\Phi\varphi = \varphi \otimes \eta$. It follows that, $\forall a \in A$, $x \in A'$,

$$\begin{aligned} \Phi'(ax) &= (\Phi'\varphi a)(\Phi' x) = (\theta\Phi\varphi a)(\Phi' x) \\ &= [\theta(\varphi \otimes \eta)(a \otimes 1)]\, \Phi' x = a\Phi' x, \end{aligned}$$

Q.E.D.

2.5. By a graded K-algebra, we mean a K-algebra graded by the natural integers.

PROPOSITION. *Let K be an algebra over the field of rational numbers. If $A = \oplus_{r \geqslant 0} A_r$ is a graded K-algebra*, then A is of the same homotopy type as A_0. If A is connected, i.e., $A_0 = K$, then A is contractible.

Proof. Construct the morphism of K-algebras $\Phi : A \to A \otimes K[t]$ such that, $\forall a_r \in A_r$, $\Phi a_r = a_r \otimes t^r$. If s_0 and s_1 are the augmentations of $K[t]$ such that $s_0 t = 0$ and $s_1 t = 1$, then $1_A = (1 \otimes s_1)\, \Phi$, and $(1 \otimes s_0)\, \Phi$ is the composition $A \xrightarrow{\rho} A_0 \xrightarrow{i} A$, where ρ is the projection and i the inclusion,

Q.E.D.

3. Homotopy of Differential Graded Algebras

3.1. Let $C = \bigoplus_{r \geqslant 0} C_r$ be a DG (differential graded) K-algebra with a differential δ. For convenience, we shall assume that the degree of δ is 1 instead of -1. If $\partial : B \to N$ is a split exact derivation, denote by $\bar{B} = \bigoplus_{r \geqslant 0} \bar{B}_r$ the DG K-algebra with $\bar{B}_0 = B$, $\bar{B}_1 = N$, $\bar{B}_r = 0$, $r \geqslant 2$, and with the differential $\bar{\delta}$ such that $\bar{\partial}\bar{B}_1 = 0$ and $\bar{\partial} b = \partial b$, $\forall b \in \bar{B}_0$. If $s : B \to K$ is an augmentation of B, denote by $\bar{s}$ the augmentation of $\bar{B}$ such that $\bar{s}\bar{B}_1 = 0$ and $\bar{s}b = sb$, $\forall b \in \bar{B}_0$.

If $C' = \bigoplus_{r \geqslant 0} C'_r$ is a DG K-algebra with a differential δ', denote by $C' \otimes \bar{B}$ the tensor product in the category of DG K-algebras. (See [*6*].) The differential of $C' \otimes B$ will be denoted by $\delta' \otimes \bar{\partial}$. Recall that

$$(C' \otimes \bar{B})_r = (C'_r \otimes \bar{B}_0) \oplus (C'_{r-1} \otimes \bar{B}_1)$$

and, $\forall x' \in C'_r$, $b \in \bar{B}$,

$$\delta' \underline{\otimes}\, \bar{\partial}(x' \otimes b) = \delta' x' \otimes b + (-1)^r x' \otimes \bar{\partial} b.$$

3.2. Definition. Two morphisms of DG K-algebras $\phi_i : C \to C'$, $i = 0, 1$, are homotopic, if there exist a commutative K-algebra B with a split exact derivation ∂, a morphism of DG K-algebras $\Phi : C \to C' \otimes \bar{B}$ and two augmentations s_0, s_1 of B such that $\phi_i = (1 \otimes \bar{s}_i)\, \Phi$, $i = 0, 1$.

Theorem. *If two morphisms of* DG *K-algebras $\phi_i : C \to C'$, $i = 0, 1$, are homotopic, then the associated chain map from the cochain complex of C to that of C' are chain homotopic.*

Proof. Let $\iota_0 : N \to B$ be the injection such that $\partial \iota_0 = 1_N$ and $s_0 \iota_0 = 0$. Then $s_1 \iota_0 \partial = s_1 - s_0$.

Let $x_r \in C_r$, $x'_r \in C'_r$, $b_0 \in \bar{B}_0$, $b_1 \in \bar{B}_1$. Let

$$\beta : C' \otimes \bar{B} \to C'$$

be the morphism of graded K-modules of degree -1 given by

$$\beta(x'_r \otimes b_0) = 0, \qquad \beta(x'_r \otimes b_1) = (-1)^r (s_1 \iota_0 b_1)\, x'_r .$$

Set $D = \beta\Phi : C \to C'$. Then

$$D\delta x_r = \beta\Phi\delta x_r = \beta(\delta' \underline{\otimes}\, \bar{\partial})\, \Phi x_r ,$$

$$\gamma' D x_r = \delta' \beta \Phi x_r .$$

Moreover,

$$\beta(\delta' \underline{\otimes} \bar{\partial})(x'_r \otimes b_0 + x'_{r-1} \otimes b_1)$$

$$= \beta[\delta' x'_r \otimes b_0 + ((-1)^r x'_r \otimes \bar{\partial} b_0 + \delta' x'_{r-1} \otimes b_1)]$$

$$= (s_1 \iota_0 \partial b_0)\, x'_r + (-1)^r (s_1 \iota_0 b_1)\, \delta' x'_{r-1}$$

$$= [(s_1 - s_0)\, b_0]\, x'_r + (-1)^r (s_1 \iota_0 b_1)\, \delta' x'_{r-1}\,,$$

and

$$\delta' \beta(x'_r \otimes b_0 + x'_{r-1} \otimes b_1) = (-1)^{r-1}(s_1 \iota_0 b_1)\, \delta' x'_{r-1}$$

so that

$$\beta(\delta' \underline{\otimes} \bar{\partial}) + \delta' \beta = 1_{C'} \otimes (\bar{s}_1 - \bar{s}_0).$$

Hence

$$D\delta + \delta' D = \phi_1 - \phi_0\,,$$

Q.E.D.

4. Homotopy Invariance of de Rham Cohomology

4.1. Let $E = E(A)$ be the Kähler module of a commutative K-algebra A, and $d = d(A) : A \to E$, the associated derivation. Denote by

$$\Lambda_A(E) = \bigoplus_{r \geqslant 0} \Lambda_A{}^r(E)$$

the exterior algebra of the A-module E. It is known [8] that the derivation

$$d : A = \Lambda_A{}^0(E) \to E = \Lambda_A{}^1(E)$$

can be extended to a differential so that the exterior algebra $\wedge_A(E)$ becomes a DG K-algebra, which gives rise to a cochain complex. The cohomology algebra $H(A)$ thus obtained will be called the de Rham cohomology algebra of A.

4.2. If $C = \oplus_{r \geqslant 0} C_r$ is a DG K-algebra, then C_0 is a K-algebra, and, for $r \geqslant 0$, C_r is a C_0-module.

Proposition. *Let $\phi^{(0)} : A = \Lambda_A{}^0(E) \to C_0$ be a morphism of K-algebras, and $\phi^{(1)} : E = \wedge_A{}^1(E) \to C_1$ a morphism of A-modules via $\phi^{(0)}$ such that $\phi^{(0)} da = \delta \phi^{(1)}$, $a \forall a \in A$. Then $\phi^{(0)}$ and $\phi^{(1)}$ can be extended uniquely to a morphism of* DG *K-algebras $\tilde{\phi} : \Lambda_A(E) \to C$.*

Proof. The uniqueness of $\tilde{\phi}$ follows from the fact that $\Lambda_A(E)$ is generated by $\Lambda_A{}^0(E)$ and $\Lambda_A{}^1(E)$.

If we regard C as an A-algebra via $\phi^{(0)}$, the morphism of A-modules

$$E \xrightarrow{\phi^{(1)}} C_1 \subset C$$

has a factorization through the tensor A-algebra $T_A(E)$:

$$E \longrightarrow T_A(E) \xrightarrow{\phi'} C.$$

Furthermore ϕ' can be factored through $\Lambda_A(E)$:

$$T_A(E) \xrightarrow{\rho} \Lambda_A(E) \xrightarrow{\tilde{\phi}} C$$

where ρ is the projection. It remains to show that $\tilde{\phi}$ commutes with the differential. We use induction on r to show that the commutativity holds on $\Lambda_A{}^r(E)$, $r \geqslant 0$. The case $r = 0$ is given. For $r \geqslant 1$, we verify that, for $x_i \in \Lambda_A{}^i(E)$ and $x_j \in \Lambda_A{}^j(E)$, $0 \leqslant i < r$, $i + j = r$,

$$\tilde{\phi} d(x_i \wedge x_j) = \tilde{\phi}(dx_i \wedge x_j + (-1)^i x_i \wedge dx_j) = \delta\tilde{\phi}(x_i \wedge x_j).$$

4.3. Write $E' = E(A')$ and $d' = d(A')$. Any morphism of K-algebras $\phi : A \to A'$ induces a morphism of K-modules $E(\phi) : E \to E'$ and therefore a morphism of DG K-algebras $\tilde{\phi} : \Lambda_A(E) \to \Lambda_{A'}(E')$. In turn, $\tilde{\phi}$ induces a morphism of graded K-algebras

$$\phi^* : H(A) \to H(A').$$

4.4. The next assertion leads to the conclusion that the de Rham cohomology functor H can be taken as a covariant functor on the homotopy category of commutative K-algebras.

THEOREM. *Let A and A' be commutative K-algebras. If two morphisms $\phi_i : A \to A'$, $i = 0, 1$, are homotopic, then $\phi_0^* = \phi_1^*$.*

Proof. Owing to Theorem 3.2, it suffices to show that, if ϕ_0 and ϕ_1 are homotopic through $\Phi : A \to A' \otimes B$ and augmentations s_0 and s_1 of B, then there exists a morphism of DG K-algebras

$$\tilde{\Phi} : \Lambda_A(E) \to \Lambda_{A'}(E') \otimes \bar{B}$$

such that $\tilde{\phi}_i = (1 \otimes \bar{s}_i)\tilde{\Phi}$, $i = 0, 1$.

We shall first construct $\tilde{\Phi}$ and then verify that $\tilde{\phi}_i = (1 \otimes \bar{s}_i)\tilde{\Phi}$. Observe that both $E' \otimes B$ and $A' \otimes N$ are $A' \otimes B$-modules and that

$$(\delta' \otimes 1_B, 1_{A'} \otimes \partial) : A' \otimes B \to (E' \otimes B) \oplus (A' \otimes N)$$

is a derivation of the K-algebra $A' \otimes B$. The composition $(\delta' \otimes 1, 1 \otimes \partial)\,\Phi$ can be taken as a derivation of A and, consequently, can be factored through the Kähler module $E : A \xrightarrow{d} E \xrightarrow{\hat{\Phi}} (E' \otimes B) \oplus (A' \otimes N)$.

Set $C = \Lambda_{A'}(E') \otimes \bar{B}$. Then $C_0 = A' \otimes B$ and

$$C_1 = (E' \otimes B) \oplus (A' \otimes N).$$

According to Proposition 4.2, $\Phi : \Lambda_A{}^0(E) \to C_0$ and $\hat{\Phi} : \Lambda_A{}^1(E) \to C_1$ can be uniquely extended to a morphism of DG K-algebras $\tilde{\Phi} : \Lambda_A(E) \to C$.

In order to verify that $\tilde{\phi}_i = (1 \otimes \bar{s}_i)\tilde{\Phi}$, observe that, for $a \in \Lambda_A{}^0(E)$,

$$(1 \otimes \bar{s}_i)\tilde{\Phi}a = (1 \otimes s_i)\,\Phi a = \phi_i a = \tilde{\phi}_i a.$$

If, for $w \in \Lambda_A{}^1(E)$, $(1 \otimes \bar{s}_i)\tilde{\Phi}w = E(\phi_i)\,w$, then $(1 \otimes \bar{s}_i)\tilde{\Phi}$ and $\tilde{\phi}_i$ coincide on $\Lambda_A{}^0(E)$ and $\Lambda_A{}^1(E)$. Proposition 4.2 implies that they must coincide on $\Lambda_A(E)$.

Denote by $\pi : (E' \otimes B) \oplus (A' \otimes N) \to E' \otimes B$ the projection. It follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "d"] & A' \otimes B \arrow[r, "1\otimes s_i"] \arrow[d, "d'\otimes 1"] & A' \arrow[d, "d'"] \\ E(A) \arrow[r, "\pi\hat{\Phi}"] & E' \otimes B \arrow[r, "1\otimes s_i"] & E' \end{tikzcd}$$

that $(1 \otimes s_i)\,\pi\hat{\Phi}d = d'\phi_i$. Since $E(\phi_i)$ is the unique morphism from the A-module E to E', which is taken as an A-module via ϕ_i, such that $E(\phi_i)\,d = d'\phi_i$, we have $E(\phi_i) = (1 \otimes s_i)\,\pi\hat{\Phi}$. Hence, for $w \in \Lambda_A{}^1(E)$,

$$(1 \otimes \bar{s}_i)\tilde{\Phi}w = (1 \otimes \bar{s}_i)\hat{\Phi}w = (1 \otimes s_i)\,\pi\hat{\Phi}w = E(\phi_i)\,w = \tilde{\phi}_i w, \qquad \text{Q.E.D.}$$

COROLLARY. *Let K be an algebra over the field of rational numbers. If $A = \oplus_{r\geqslant 0} A_r$ is a commutative graded K-algebra, then $H(A) \simeq H(A_0)$. In particular, if A is a polynomial K-algebra, then $H(A) \simeq H(K)$.*

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